Low Grade Matrices and Matrix Fraction Representaions

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Abstract. The matrix equation $MA = N$ is considered. The lgrade of a matrix A is the the largest rank of any subdiagonal block of a symmetric partition of a square matrix. When A has lgrade d, representation results are given for $MA = N$ with M and N having lower bandwidth d. M can be chosen to be lower triangular or unitary. A second result is that if $MA = N$ with M and N have lower bandwidths d_M and d_N then $A = U + V$ with the $lwidth(U) = d_N - d_M$ and U is an upper triangular matrix.

1. Introduction

The special structure of inverses of band matrices is well understood and clearly exposited in [2, 3, 4, 7, 8]. We generalize these results to the equation $MA = N$ where M and N have bandwidth restrictions. When N is the identity matrix, our results apply to the standard case of band matrices and Green matrices.

We now give our basic definitions of bandwidth and grade.

Definition 1.1. *An $n \times n$ matrix M is called* lower banded with lower bandwidth *(lwidth) d if $M_{ij} = 0$ for $i > j + d$. M is said to have strict lower bandwidth d if $M_{j+d,j} \neq 0$ for $1 \leq j \leq n - d$.*

A matrix T is upper triangular if and only if lwidth $T \leq 0$. A matrix with upper bandwidth 1 is called lower Hessenberg matrix, and a matrix with lower bandwidth 1 is called an upper Hessenberg matrix. We denote the Frobenius norm by $\|\cdot\|_F$ and the lower shift matrix by Z. The unit vector in the kth coordinate is denoted e_k.

We define the lower grade of a square matrix as the maximum rank of the lower subdiagonal block.

Definition 1.2. *A partition of a matrix*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \tag{1.1}$$

is called symmetric *if X and W are square.*

If a symmetric partition exists for M, then M is square. In this article, we will reserve the symbols X, Y, Z and W to denote the subblocks of a symmetric partition of M.

Definition 1.3. *The upper (lower) grade of a matrix M, written* $\mathrm{ugrade}(M)$ $(\mathrm{lgrade}(M))$ *is the maximum rank of a part of a symmetric partition above (below) the diagonal. The* grade *of a matrix M is the maximum rank of an off diagonal part of a symmetric partition, that is,* $\mathrm{grade}(M) = \max\{\mathrm{lgrade}(M), \mathrm{ugrade}(M)\}$.

Proposition 1.1. *Companion matrices and Jordan matrices have grade 1. Elementary row (column) operation matrices, Householder, and hyperbolic Householder transformations, Givens and signed Givens rotations all have grade 1.*

Our definition of symmetric partition matches that of Rózsa, Romani, Bevilacqua [9]. Their focus is on persymmetric partitions while we derive results for the grade corresponding to symmetric partitions. In the next section, we give properties of matrix grade. Sections 3 and 4 describe some special classes of low grade matrices: consecutive subblock products and triangular input bounded matrices. Section 5 gives matrix fraction representation for low grade matrices. Section 6 gives a decompostion $A = U + V$ where V is low rank and U has prescribed lwidth under the hypothesis that $MA = N$ where M and N have prescribed lwidths. Section 7 show that any matrix M of $\mathrm{lgrade} = d$ may be approximately decomposed $M = U + V$ with V is of rank d and upper triangular U.

2. MATRIX GRADE

We establish some properties of the grade and lgrade. Analogous results hold for ugrade. Some of the proofs are in the Appendix.

Theorem 2.1 [Basic Properties]**.** *Let M is $n \times n$ matrix:*
i) $\mathrm{grade}(M) \le \left\lfloor \frac{n}{2} \right\rfloor$.
ii) $\mathrm{grade}(M) \le \mathrm{rank}(M)$.
iii) $\mathrm{lgrade}(M) \le \mathrm{lwidth}(M)$,*and* $\mathrm{grade}(M) \le \mathrm{width}(M)$.
iv) $\mathrm{lgrade}(M^*) = \mathrm{ugrade}(M)$ *and* $\mathrm{grade}(M^*) = \mathrm{grade}(M)$.
v) $\mathrm{ugrade}(L) = 0$ *is the same as L lower triangular, and* $\mathrm{lgrade}(U) = 0$ *is the same as U upper triangular.* $\mathrm{grade}(M) = 0$ *is the same as M diagonal.*
vi) Let N be a minor of M. Then $\mathrm{lgrade}(N) \le \mathrm{lgrade}(M)$. *Similarly for* grade.
vii) $\mathrm{lgrade}(M_1 \oplus M_2) = \max\{\mathrm{lgrade}(M_1), \mathrm{lgrade}(M_2)\}$ *for square M_1 and M_2.*

This definition of matrix grade is subadditive and submultiplicative:

Theorem 2.2.

$$\text{lgrade}(M_1 + M_2) \leq \text{lgrade}(M_1) + \text{lgrade}(M_2) \tag{2.1}$$
$$\text{lgrade}(M_1 M_2) \leq \text{lgrade}(M_1) + \text{lgrade}(M_2). \tag{2.2}$$

Similarly for ugrade *and* grade.

The following "reduction lemma", lets us find a succesion of matrices which preserve lgrade while transforming M.

Lemma 2.3. *Let M and G be $n \times n$ matrices with* lgrade$(M) = d$. *If for all symmetric partitions of M there exists a matrix C_i such that the corresponding partition of GM satisfies $(GM)_{12} = C_i M_{12}$, then* lgrade$(GM) \leq d$. *Only partitions with column dimension $(M_{12}) > d$ and row dimension$(M_{12}) > d$ need be considered.*

Clearly $C_i M_{12}$ has rank less than $(d+1)$. Typically, G is chosen to be a "consecutive subblock product" as defined in section 3. We now show that the set of matrices of lgrade $= d$ is closed.

Lemma 2.4. *Let $\|M_k - M\|_F \to 0$ as $k \to \infty$, and $\lim_{k\to\infty}$* lgrade$(M_k) = d$. *Then* lgrade$(M) \leq d$. *Similarly for* ugrade *and* grade.

Choose any symmetric partition:

$$M_k = \begin{pmatrix} X_k & Y_k \\ Z_k & W_k \end{pmatrix} \tag{2.3}$$

then $\lim_{k\to\infty} \|Z - Z_k\|_F = 0$ and rank$(Z) \leq d$ by continuity of the singular values of Z in the Frobenius norm. Similarly for rank(Y_k). ■

Since any other matrix norm is equivalent to the Frobenius norm, this result holds for all matrix norms. As shown in [9], the grade of M^{-1} equals the grade of M:

Theorem 2.5. *[9] For any square invertible matrix M,*

$$\text{lgrade}\left(M^{-1}\right) = \text{lgrade}(M), \tag{2.4}$$
$$\text{grade}\left(M^{-1}\right) = \text{grade}(M). \tag{2.5}$$

Proof: Choose any symmetric partition of M as in 1.1 and conformably partition

$$M^{-1} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{2.6}$$

so $RX + SZ = 0$. If X is nonsingular, then $\operatorname{rank}(R) \leq \operatorname{rank}(Z) \leq \operatorname{lgrade}(M)$. This gives $\operatorname{lgrade}(M^{-1}) \leq \operatorname{lgrade}(M)$, but we may equally apply the argument to M^{-1}, thus $\operatorname{lgrade}(M^{-1}) = \operatorname{lgrade}(M)$. If X is singular, we obtain the result by adding a small multiple of the identity to M and taking a limit. The result for ugrade follows by applying this result to M^* and this establishes the result for grade. ∎

Corollary 2.6. $\operatorname{lgrade}(M^{-1}) \leq \operatorname{lwidth}(M)$, *and* $\operatorname{grade}(M^{-1}) \leq \operatorname{width}(M)$.

In particular, bidiagonal matrices and their inverses have grade 1.

A consequence of submultiplicativity is

Lemma 2.7 [triangular factor grade]**.** *Let $M = NU$ for upper triangular U. Then* $\operatorname{lgrade}(M) \leq \operatorname{lgrade}(N)$, *and if either U or M has nonsingular principal minors then* $\operatorname{lgrade}(M) = \operatorname{lgrade}(N)$.

Note that M nonsingular implies that U is nonsingular and has nonsingular principal minors.

Corollary 2.8. *Let $M = LR$ for lower triangular L and upper triangular R. Then* $\operatorname{lgrade}(M) \leq \operatorname{lgrade}(L)$ *and* $\operatorname{ugrade}(M) \leq \operatorname{ugrade}(R)$. *Additionally, if either L or R is nonsingular,* $\operatorname{lgrade}(M) = \operatorname{lgrade}(R)$ *and* $\operatorname{ugrade}(M) = \operatorname{ugrade}(R)$.

Proof: Apply Lemma 2.7 to M and M^*. ∎

3. CONSECUTIVE SUBBLOCK PRODUCTS

As mentioned earlier, many of the basic building blocks for matrix analysis such as elementary row and column operation matrices, signed Givens rotations, and Householder transformations have grade 1. We now present another class of low grade matrices, which we call consecutive subblock products (CSP):

Definition 3.1. *Let F_k be an $n \times n$ matrix such that $F_k e_j = e_j$ and $e_j^* F_j = e_j^*$ for $j < k$ and for $j > k + d$. Then $M = F_1 \cdots F_{n-d}$ is called a consecutive subblock product of order d.*

F_k is the identity except for a $(d+1) \times (d+1)$ block on the diagonal in rows (and columns) k through $k+d$. This block can be chosen arbitrarily. If the F_k are unitary (orthogonal) then so is the product, similarly if the F_k are all lower (upper) triangular, then so is the product.

For the $d = 2$ case our results are summarized by

Proposition 3.1 [[1]]**.** *A product of $n-1$ consecutive Givens rotations is a unitary Hessenberg matrix, and by the LQ decomposition, a unitary Hessenberg matrix is a product of $n-1$ consecutive Givens rotations.*

Proposition 3.1 follows from ii) and iii) of Theorem 3.2. This representation is extensively studied in [1]

Theorem 3.2 [Consecutive Subblock Product Properties]**.** *Let $M = F_1 \cdots F_{n-d}$ be a consecutive subblock product of order d.*
i) If each F_k is nonsingular, then $M^{-*} = (F_1)^{-*} \cdots (F_{n-d})^{-*}$.
ii) $\operatorname{lwidth} M \leq d$*. If for each k the element $(F_k)_{k+d,k}$ is nonzero then the consecutive subblock product has strict lower width d.*
iii) $\operatorname{ugrade} M \leq d$ *and* $\operatorname{grade} M \leq d$*.*

Proof: ii) is proven by induction. The proof of iii) is in the appendix.

4. HYPEREXCHANGE AND TRIANGULAR INPUT BOUNDED MATRICES

A generalization of unitary and S-unitary matrices is hyperexchange matrices:

Definition 4.1. *A matrix is hyperexhange if and only there are signature matrices S_1 and S_2 such that $MS_1M^* = S_2$.*

Theorem 4.1. *Let M be hyperexchange, $MS_1M^* = S_2$ with S_1 and S_2 signature matrices. Then* $\operatorname{lgrade}(M) = \operatorname{ugrade}(M)$*.*

Proof: Since $M^{-1} = S_1M^*S_2$, and $\operatorname{grade}(S_1) = \operatorname{grade}(S_2) = 0$, then $\operatorname{lgrade}(M) = \operatorname{lgrade}(M^{-1}) = \operatorname{ugrade}(M)$. ■

The $d = 2$ case is an approximate converse of Proposition 3.1:

Corollary 4.2. *Let M be hyperexchange and Hessenberg. Then* $\operatorname{grade}(M) = 1$*.*

As described in [5], triangular input balanced (TIB) system are useful for system identification. In [6], some TIB systems are analyzed and a matrix fraction representation is given. We now show that TIB systems have are low-grade.

Definition 4.2. *A matrix pair is (A, B) is a lower triangular input balanced system if and only if*

$$AA^* + BB^* = \mathbb{I} \ , \tag{4.1}$$

where A is a $n \times n$ lower triangular, B is size $n \times d$, and $\mathbb{I}$ is the identity matrix.

Theorem 4.3. *Let (A, B) be a lower TIB system with B of full column rank. Then* $\operatorname{grade}(A) \leq \operatorname{rank}(B)$*.*

Proof: There are Z and W such that $M = \begin{pmatrix} A & B \\ Z & W \end{pmatrix}$ is $(n+d)\times(n+d)$ and U is unitary. The result follows from the following lemma. ■

Lemma 4.4. *Suppose that X is lower triangular where X is the upper left submatrix in a symmetric partition of M, as in Eq. (1.2). Then* $\mathrm{grade}\,(Y) \le \text{number of columns}\,(B)$.

Sections 5 -7 now present general representation results for low grade matrices.

5. BAND FRACTION REPRESENTATIONS

In this section, we show that if $\mathrm{lgrade}\, M \le d$ then then are matrices G and H such that $GM = H$ where G and H have band structure. If M satisfies a generic condition, G is invertible and we have a structured matrix fraction representation.

Theorem 5.1. *Suppose* $\mathrm{lgrade}\, M \le d$. *Then there are matrices L and H such that $LM = H$ with L is lower triangular and* $\mathrm{lwidth}\, L \le d$ *and* $\mathrm{lwidth}\, H \le d$ *and* $\mathrm{uwidth}\, H \le \mathrm{uwidth}\, M$.

The proofs in this section are inductive and begin by partioning M as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{5.1}$$

where M_{11} is $(d+1)\times(d+1)$. Since $\mathrm{lgrade}\, M \le d$, then $\mathrm{rank}\, M_{21} \le d$. A matrix is given which eliminates the last row of M_{21} while preserving the lgrade of the product.

Proof: The result is trivially true for $n \le d$. Suppose it is true for $n-1$. Partition M as in (5.1) Since $\mathrm{lgrade}\, M \le d$, then $\mathrm{rank}\, M_{21} \le d$ and there is a $(d+1)\times(d+1)$ lower triangular matrix F_{n-d-1} such that the last row of $F_{n-d-1}M_{21}$ vanishes. (For example,. there exists a nonzero $(d+1)$ vector α such that $\alpha^* M_{21} = 0$, let F_{n-d-1} be any lower triangular matrix with last row α^*.) By the Lemma 2.3, the rank of the first d rows of $F_{n-d-1}M_{21}$ have rank less than or equal to d. This reduces the problem to the claimed factorization of the $(n-1)\times(n-1)$ principle minor of M, from which one determines F_{n-d-2}, ..., F_1, then put $L = F_1 \cdots F_{n-d-1}$. ■

Corollary 5.2. *In Theorem 5.1, L may be chosen to be a consecutive subblock product with $F_i = I_n + \mathbf{e}_{i+d}(\alpha_i - \mathbf{e}_{i+d})^*$ where $\mathbf{e}_{i+d}$ is unit vector in the $i+d$ th coordinate.*

Suppose in the last proof, that all the requisite vectors α can be taken to have a nonzero last element, then L is nonsingular and $M = L^{-1}H$. The following condition guarantees the existence of such α_i:

Definition 5.1 [Condition(G)]. *Let M be a $n \times n$ matrix and define the $(d+1) \times (d+1)$ submatrices M_i by $M_i \equiv M_{i:(i+d),1:(d+1)}$ for $1 \le i < n-d$. The matrix M satisfies condition (G) if and only if for all i with $1 \le i < n-d$, $\mathbf{e}_{d+1} \notin \mathrm{Range}(M_i)$, where the $d+1$-vector $\mathbf{e}_{d+1} \equiv (0, 0 \ldots 1)^*$.*

Condition (G) implies that for each i there exists a nonnull $(d+1)$-vector α_i such that $\alpha_i^* M_i = 0$ and $\alpha_{i,d+1} \neq 0$.

Corollary 5.3. *If $\mathrm{lgrade}\, M \le d$ and M satisfies Condition G, then L is nonsingular and $M = L^{-1}H$ where L and H are given in Theorem 5.1. F_k and L may be chosen to be unit lower triangular.*

Theorem 5.4. *Suppose $\mathrm{lgrade}\, M \le d$. There are matrices Q and H such that $QM = H$ where Q is unitary consecutive subblock product (and real orthogonal if M is real) and $\mathrm{lwidth}\, Q \le d$ and $\mathrm{lwidth}\, H \le d$ and*

Proof: Repeat the proof of Theorem 5.1 except this time choose F_{n-d-1} to be the Householder reflection determined $F_{n-d-1}\alpha = e_{d+1}$. ∎

Theorem 5.5. *Suppose $\mathrm{lgrade}\, M \le d$. Then there are matrices Q and H such that $MQ = H$ with Q is an unitary consecutive subblock product of width d (and real orthogonal if M is real) and $\mathrm{lwidth}\, Q \le d$ and $\mathrm{lwidth}\, H \le d$.*

Proof: The result is trivially true for $n \le d$. Suppose it is true for $n-1$. There is a $(d+1) \times (d+1)$ unitary matrix F_1 such that the first column of $M_{21}F_1$ vanishes. (For example, there exists a nonzero vector α such that $M_{21}\alpha = 0$, let F_1 be the Householder reflection such that $F_1\alpha = e_1$.) Trivially, the rank of the first d rows of $M_{21}F$ have rank less than or equal to d, and this reduces the problem to the $(n-1) \times (n-1)$ submatrix of M below the first row and column, which determines $F_2, ..., F_{n-d-1}$. Put $Q = F_1 \cdots F_{n-d-1}$. ∎

Transposing the theorems and proofs gives similar results for upper graded matrices. In the constructions of Theorems 5.4 and 5.5, F_k can be a product of d Givens rotations instead of a Householder reflection.

6. $M = U + V$ DECOMPOSITION WITH LOW LWIDTH U AND LOW RANK V

The inverse of a strict invertible bidiagonal matrix, M, is the sum of a matrix with lwidth$= -1$ and a rank-1 matrix [2, 3, 4, 7, 8]. In our notation, these authors study, A where $MA = \mathbb{I}$ and M has low rank. We now study the generalized problem: $MA = N$.

Theorem 6.1 [Band Ratio Subtriangle]**.** *Suppose* lwidth $(M) = d_M$ *and* lwidth $(N) = d_N$, *and*

$$MA = N. \tag{6.1}$$

Then $A = U + V$ *where* lwidth $U = d_N - d_M$ *and* rank $V = \dim\ker T$, *where* T *is the upper triangular matrix*

$$T = \begin{cases} Z^{*d_M} M & d_M \geq 0 \\ Z^{(-d_M)} M & d_M < 0 \end{cases}. \tag{6.2}$$

The proof is in the Appendix. In [6], we give, an explicit matrix fraction representation of TIB systems with $d = 1$, $A = M^{-1}N$, where M and N are bidiagonal lower triangular matrices.

Corollary 6.2. *Let* M *be a nonsingular strict consecutive subblock product of order* d*. then* $M = U + V$, *where* lwidth $U \leq -d$ *and* rank $V \leq d$*.*

The requirment that M is strict is necessary as the following example illustrates. Set $M = A = N = \mathbb{I}$. M is a nonsingular nonstrict consecutive subblock product of order d for any $d > 0$ but is not of the form $M = U + V$ where lwidth $U \leq -d$ and rank $V \leq d$.

Combining Theorem 6.1 with Theorem 5.4 yields

Corollary 6.3. *Suppose* lgrade $M \leq d$*. Then there are unitary* Q *with* lwidth $(Q) = d$ *and* H *with* lwidth $(H) = d$, *such that* $QM = H$ *and* $M = U + V$ *where* U, *is upper triangular,* rank $V = \dim\ker T$, *where* $T = Z^{*d}Q$*. If* M *has strict* lwidth d, *then* $\dim\ker T \leq d$*.*

Proof: Apply Theorem 6.1 to the factorization $QM = H$ given by theorem 5.4. ■

7. Approximate $U + V$ Decomposition for Small Grade Matrices

We now show that the $U + V$ decompostion can be achieved to any accuracy for an arbitrary low grade matrix.

Theorem 7.1. *Suppose* M *is* $n \times n$ *and* lgrade $(M) = d$*. Then for any* $\varepsilon > 0$ *there is an upper triangular* U_e *and* V_ε *of rank* d *such that* $\|M - (U_\varepsilon + V_\varepsilon)\| < \varepsilon$*.*

The following lemma allows us to paste together two overlapping low rank matrices while only alterring the upper corner.

Lemma 7.2 [The 'pasting' lemma]**.** *. Suppose* M *is a matrix*

$$M = \begin{pmatrix} x^* & y \\ Z & w \end{pmatrix} \tag{7.1}$$

where

$$\operatorname{rank}\begin{pmatrix} Z & w \end{pmatrix} \leq \operatorname{rank}\begin{pmatrix} x^* \\ Z \end{pmatrix} = d. \tag{7.2}$$

Then for any $\varepsilon > 0$ there are scalars γ and δ such that

$$\operatorname{rank}\begin{pmatrix} x^* & \delta \\ Z + \gamma w x^* & w \end{pmatrix} = d \tag{7.3}$$

and $|\gamma| \, \|wx^*\|_F < \varepsilon$. In particular

$$\operatorname{rank}\begin{pmatrix} Z + \gamma w x^* & w \end{pmatrix} = \operatorname{rank}\begin{pmatrix} Z & w \end{pmatrix} \leq d \tag{7.4}$$

$$\operatorname{rank}\begin{pmatrix} x^* \\ Z + \gamma w x^* \end{pmatrix} = \operatorname{rank}\begin{pmatrix} x^* \\ Z \end{pmatrix} = d. \tag{7.5}$$

Proof of Theorem 7.1:

Apply the pasting lemma to each subblock with upper right corner on the main diagonal. Next apply the pasting lemma to each subblock with upper right corner on the first superdiagonal. The pasting lemma is applied to each successive superdiagonal. ■

8. Summary

We have established several important structure results for graded matrices. The matrix fraction decompositions of Section: $M = L^{-1}H$ and $M = Q^*H$ generalized the results in [6] and show that low grade matrices have a special structure. Both L and Q are consecutive subblock products.

Any matrix M with $\operatorname{lgrade}(M) = d$ may be approximated to arbitrary accuracy as the sum: $M = U + V$ where U is upper triangular U and V is rank d. When $MA = N$ and M and N have small lower bandwidth, the decomposition $A = U + V$ is exact wherethe lwidth of U and the rank of V are given in Theorem 6.1.

9. Appendix: Proofs

Proof of Theorem 2.1:

Part iii) Choose any symmetric partition as in 1.1. Z is upper triangular, with at most $\operatorname{width}(M)$ nonzero columns, so $\operatorname{rank}(Z) \leq \operatorname{width}(M)$, similarly for Y.

Part vi) Let Z be a subdiagonal part of a symmetric partition of N of maximal rank. Since N is obtained by deleting a row and column of M, either Z, or Z with a column or row adjoined, is a subdiagonal part of a symmetric partition of M. Hence $\operatorname{lgrade}(N) \leq \operatorname{rank}(Z) \leq \operatorname{lgrade}(M)$.

Proof of Part vii): Let

$$M_1 \oplus M_2 = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \tag{9.1}$$

be any symmetric partition of $M_1 \oplus M_2$. Since

$$M_1 \oplus M_2 = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_2 \end{pmatrix} \tag{9.2}$$

then either $Z = \begin{pmatrix} Z_1 \\ 0 \end{pmatrix}$ where Z_1 is a subdiagonal block of M_1 or $Z = \begin{pmatrix} 0 & Z_2 \end{pmatrix}$ where Z_2 is a subdiagonal block of M_2, hence

$$\operatorname{lgrade}(M_1 \oplus M_2) \le \max\{\operatorname{lgrade}(M_1), \operatorname{lgrade}(M_2)\}, \tag{9.3}$$

but both M_1 and M_2 are minors of $M_1 \oplus M_2$ so

$$\operatorname{lgrade}(M_1 \oplus M_2) \ge \max\{\operatorname{lgrade}(M_1), \operatorname{lgrade}(M_2)\} . \tag{9.4}$$

■

Proof of Theorem 2.2: Choose any symmetric partition

$$M_i = \begin{pmatrix} X_i & Y_i \\ Z_i & W_i \end{pmatrix}. \tag{9.5}$$

Then

$$M_1 + M_2 = \begin{pmatrix} X_1 + X_2 & Y_1 + Y_2 \\ Z_1 + Z_2 & W_1 + W_2 \end{pmatrix} \tag{9.6}$$

and $\operatorname{rank}(Z_1 + Z_2) \le \operatorname{lgrade}(M_1) + \operatorname{lgrade}(M_2)$.

Since

$$M_1 M_2 = \begin{pmatrix} X_1X_2 + Y_1Z_2 & X_1Y_2 + Y_1W_2 \\ Z_1X_2 + W_1Z_2 & Z_1Y_2 + W_1W_2 \end{pmatrix} \tag{9.7}$$

and $\operatorname{rank}(Z_i) \le d_i$, we have $\operatorname{rank}(Z_1X_2 + W_1Z_2) \le \operatorname{lgrade}(M_1) + \operatorname{lgrade}(M_2)$, hence $\operatorname{lgrade}(M_1M_2) \le \operatorname{lgrade}(M_1) + \operatorname{lgrade}(M_2)$. ■

Proof of Lemma 2.7: $\operatorname{lgrade}(M) \le \operatorname{lgrade}(N) + \operatorname{lgrade}(U) = \operatorname{lgrade}(N)$. Choose a symmetric partition

$$N = \begin{pmatrix} X_N & Y_N \\ Z_N & W_N \end{pmatrix} \text{ and } U = \begin{pmatrix} X_U & Y_U \\ 0 & W_U \end{pmatrix}, \tag{9.8}$$

so that Z_N has maximal rank, that is, $\operatorname{rank}(Z_N) = \operatorname{lgrade}(N)$. Then the corresponding partition of M is

$$M = \begin{pmatrix} X_NX_U & X_NY_U + Y_NW_U \\ Z_NX_U & Z_NY_U + W_NW_U \end{pmatrix}. \tag{9.9}$$

If U has nonsingular principal minors, then X_U is nonsingular and so $\operatorname{lgrade}(N) = \operatorname{rank}(Z_N) \leq \operatorname{lgrade}(M)$. Now if M has nonsingular principal minors then $X_N X_U$ is nonsingular, hence X_U is nonsingular. ■

Proof of iii) of Theorem 3.2:

By induction. It is clearly true for $n = d+1$. Suppose then, that the result is true for $(n-1) \times (n-1)$ consecutive subblock products of order d. Now choose any symmetric partition of M as in (1.2). We decompose M as the product of F_1 with $F_2 F_3 \ldots F_{n-d}$ and denote the analogous symmetric partition of $F_2 F_3 \ldots F_{n-d}$ by $\{X_2, Y_2, Z_2, W_2\}$:

$$M = \begin{pmatrix} \Phi \oplus \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} X_2 & Y_2 \\ Z_2 & W_2 \end{pmatrix} = \begin{pmatrix} \Phi \oplus \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & M_2 \end{pmatrix}. \tag{9.10}$$

where Φ is the $(d+1) \times (d+1)$ principal subminor of F_1, and $\Phi \oplus \mathbb{I}$ conforms to X_2. Here M_2 is an $(n-1) \times (n-1)$ consecutive subblock product of order d, we have $\operatorname{grade} M_2 \leq d$. Now $\operatorname{grade}(1 \oplus M_2) \leq d$, so $\operatorname{rank} Y_2 \leq d$

$$\operatorname{rank} Y = \operatorname{rank}(\Phi \oplus \mathbb{I}) Y_2 \leq \operatorname{rank} Y_2 \leq d. \tag{9.11}$$

The result follows from the reduction lemma. ■

Proof of Theorem 6.1: Suppose $d_M \geq 0$. Then

$$Z^{*d_M} M A = T A = Z^{*d_M} N \tag{9.12}$$

for the upper triangular matrix $T = Z^{*d_M} M$. Let $R = T + W$ be nonsingular upper triangular with $\operatorname{rank} W = \dim \ker T$, then

$$A = R^{-1} T A + R^{-1} W A = R^{-1} Z^{*d_M} N + R^{-1} W A\ . \tag{9.13}$$

Define $U = R^{-1} Z^{*d_M} N$ and $V = R^{-1} W A$. Since $\operatorname{lwidth} R^{-1} Z^{*d_M} N \leq d_N - d_M$, we have $A = U + V$ where $\operatorname{lwidth} U \leq d_N - d_M$ and $\operatorname{rank} V \leq \dim \ker T$.

If $d_M < 0$, modify this proof by replacing Z^{*d_M} with Z^{-d_M} everywhere in the last paragraph.

Proof of Lemma 7.2:

Suppose $w = Zu$, then put $\gamma = 0$ and $\delta = x^* u$, and verify

$$\begin{pmatrix} x^* & \delta \\ Z & w \end{pmatrix} = \begin{pmatrix} x^* \\ Z \end{pmatrix} \begin{pmatrix} \mathbb{I} & u \end{pmatrix}. \tag{9.14}$$

Alternatively, suppose w is not in the span of the columns of Z. Then $\operatorname{rank} Z < d$, but $\operatorname{rank} \begin{pmatrix} x^* \\ Z \end{pmatrix} = d$. Z has at least d columns, and let u be the component of x perpendicular to the span of the rows of Z. Note that $w \neq 0$ and $u \neq 0$. Put $\gamma = (2\|wx^*\|_F)^{-1}\varepsilon$, $\delta = \gamma^{-1}x^*u = \gamma^{-1}u^*u$, and verify

$$\begin{pmatrix} x^* & \delta \\ Z+\gamma wx^* & w \end{pmatrix} = \begin{pmatrix} x^* \\ Z+\gamma wx^* \end{pmatrix} \begin{pmatrix} \mathbb{I} & \gamma^{-1}u \end{pmatrix}. \tag{9.15}$$

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