Band Matrix Representation of Triangular Input Balanced Form

Andrew P. Mullhaupt and Kurt S. Riedel
Courant Institute of Mathematical Sciences
New York University
New York, NY 10012-1185

October 23, 1998

Abstract

For generic lower triangular matrices, A, we prove that $A_{ij} = \sum_{q=1}^{d} H_{iq}\bar{G}_{jq}$ for $i > j$ is equivalent to $A = M^{-1}N$ where M and N are $d+1$ banded matrices. A lower triangular matrix A is input balanced of order/rank d if there exists a rank-d matrix B such that $AA^* = I - BB^*$. We prove that if A is triangular input balanced then generically, $A = M^{-1}N$ where M and N are $d+1$ banded matrices. This also implies $A_{i,j} = \sum_{q=1}^{d} H_{iq}\bar{G}_{jq}$ for $i > j$. When B is a vector, explict representations are given and the eigendecomposition is evaluated.

1 Introduction

We consider systems (A, B) where A is a $n \times n$ matrix and B is a $n \times d$ matrix with $n \geq d$. We say that a system, (A, B), or a matrix, B, is true rank d when $d =$ rank $B=$ column dimension of B. The system is in lower triangular input balanced (LTIB) form if and only if

$$AA^* = I - BB^* \ , \tag{1.1}$$

with A *lower triangular* and I is the identity matrix. When both A and B are lower triangular, we say that the system is $LLTIB$. We restrict our attention to LTIB systems of true order/rank$-d$. The advantages of using LTIB form for system identification are described in [7]. In this note, we derive two representations of LTIB form, the first in terms of band matrices and the second as the sum of a diagonal matrix and the strictly lower triangular (LT) part of a rank-d matrix. In the single input case ($d = 1$), we relate these representations to the orthonormal bases functions of [8]. For this case, we derive the eigenvectors of A and explicit representations of (A, B) are given.

In [6], we proved that the $d = 1$ case has the representation $A_{ik} = b_i \bar{g}_k$ for $i > k$. This representation allows fast matrix vector multiplication and fast solution of matrix equations: $A\mathbf{x} = \mathbf{c}$. In this note, we prove the following representation results for arbitrary rank LTIB systems:

Theorem 1.1 *Let (A, B) be a LLTIB system of true order/rank d. There exist $n \times n$ banded LT matrices M and N of bandwidth $(d+1)$ such that $MA = N$ and $MB = C$ where C is a $n \times d$ LT matrix with $C_{ik} = 0$ for $i > d$. If B satisfies condition (*), then M is invertible and $A = M^{-1}N$.*

Condition (*) is defined below and is generically satisfied.

Theorem 1.2 *Let M and N be $n \times n$ banded LT matrices of bandwidth $d + 1$. If M is invertible with $M_{i+d,i} \neq 0$ for $1 \leq i \leq n - d$, then*

$$\left(M^{-1}N\right)_{ik} = \sum_{j=1}^{d} H_{ij}\bar{G}_{kj} \qquad \text{for} \qquad i > k \tag{1.2}$$

for some $n \times d$ matrices H and G.

Corollary 1.3 *Let (A, B) be a LTIB system of true order/rank d with B satisfying condition (*). There exist $n \times d$ matrices H and G such that $A_{ik} = (HG^*)_{ik}$.*

We also prove a pseudo-converse of Theorem 1.2:

Definition 1.4 *A $n \times n$ LT matrix A is of type diagonal plus lower rank d if and only if $A_{ik} = (HG^*)_{ik}$* for *$i > k$ for some $n \times d$ matrices H and G.*

The representation of the off-diagonal terms is not unique since if (H, G) represents the off-diagonal terms then so does (HT, GT^{-*}) for any invertible $d \times d$ matrix T.

Postmultiply B by a unitary matrix to transform B to triangular form.

Theorem 1.5 *Let A be a $n \times n$ LT matrix of type diagonal plus lower rank d: $A_{ik} = (HG^*)_{ik}$* for *$i > k$. If H satisfies condition (*), then there exist $n \times n$ banded LT matrices, M and N, of bandwidth $d + 1$ such that $A = M^{-1}N$.*

For both Theorem 1.1 and 1.5, we impose the following condition:

Definition 1.6 *Let B be a $n \times d$ matrix and define the $(d+1) \times d$ matrices B_i by $B_i \equiv B_{i-d-1:i,(\cdot)}$ for $d+1 < i \leq n$, where $B_{i-d-1:i,(\cdot)}$ is the $(d+1) \times d$ matrix that consist of the $d+1$ rows of B from the $(i-d-1)$-th row to the i-th row. The matrix B satisfies condition (*) if and only if for all i with $d+1 < i \leq n$, $\mathbf{e}_{d+1} \notin \mathrm{Range}(B_i)$, where the $d+1$-vector $\mathbf{e}_{d+1} \equiv (0, 0 \ldots 1)^*$.*

Condition (*) implies that for each i there exists a nonnull $(d+1)$-vector $\mathbf{v}_i$ such that $\mathbf{v}_i^* B_i = 0$ and $v_{i,d+1} \neq 0$.

We now present a variant of Theorem 1.1 that is proved using different methods:

Theorem 1.7 *Let (A, B) satisfy $D - ADA^* = BB^*$ where D is a diagonal, positive definite matrix and A is LT and B is a $n \times d$ matrix of rank d. If $(B|A)$ have nonvanishing principal subminors, then A has the band fraction representation: $A = M^{-1}N$ where M and N are $n \times n$ banded LT matrices M and N of bandwidth $(d+1)$.*

Proof of Theorem 1.7:

By Theorem 3.2.1 of [3], $(B|A)$ has a $L^{-1}R$ representation. We denote by $\tilde{R}$ the submatrix of R containing columns $(d+1)$ through $(n+d)$. Since A is LT, $\tilde{R} = LA$ is LT of bandwidth $(d+1)$. Note $R(I_d \oplus D)R^* = LDL^*$. By Theorem 4.3.1 of [3], L has bandwidth $(d+1)$.

■

2 Proofs

To prove Theorem 1.2, we review some results on the inverses of banded matrices. For the case of lower triangular matrices, the results of [1, 4, 9] simplify:

Definition 2.1 *A matrix M is a strict $\{0, d\}$ band matrix if and only if $M_{ij} = 0$ for $i < j$, $M_{ij} = 0$ for $i > j + d$ and $M_{ii} \neq 0$, $M_{i+d,i} \neq 0$.*

Definition 2.2 *A matrix M is of strict rank $\{0, d\}$ if and only if $M_{ij} = 0$ for $i < j$,*

$$M_{ij} = (HG^*)_{ij} \quad \text{for} \quad i + d > j \tag{2.1}$$

and $M_{ii} \neq 0$, $(HG^*)_{i+d,i} \neq 0$.

Definition 2.2 corresponds to a $\{0,d\}$ semiseparable matrix in the terminology of [9]. This definition implies that $(HG^*)_{i,i+k} = 0$ for $1 \le k < d$. This condition imposes $\sum_{i=1}^{d-1}(n-i) = n(d-1) - d(d-1)/2$ constraints. If (H,G) represents the off-diagonal of a strict rank $\{0,d\}$ matrix, then so does (HT, GT^{-*}) for any invertible $d \times d$ matrix T. Thus the class of strict rank $\{0,d\}$ matrices has dimension $2nd - d^2 - n(d-1) + d(d-1)/2 = n(d+1) - d(d+1)/2$.

Our proof of Theorem 1.2 is based on

Theorem 2.3 ([1, 9]) *A nonsingular matrix M is a strict $\{0,d\}$ matrix if and only if its inverse has strict rank $\{0,d\}$.*

Proof of Theorem 1.2: By hypothesis, M satisfies the hypothesis of Theorem 2.3 so it has a representation given by (2.1):

$$\left(M^{-1}N\right)_{ik} = \sum_{j=k}^{\min\{n,k+d\}} \left(\sum_{q=1}^{d} H_{iq}\bar{G}_{jq}\right) N_{jk} = \sum_{q=1}^{d} H_{iq} \left(\sum_{j=k}^{\min\{n,k+d\}} \bar{G}_{jq}N_{jk}\right) \quad \text{for} \quad i > k \ . \tag{2.2}$$

In adding terms to the summation, we use the property of semisimple LT matrices that is noted below Definition 2.2. ■

The analogous proof shows that NM^{-1} is also of the type diagonal plus strict rank-d under the hypotheses of Theorem 1.2.

Proof of Theorem 1.5:

We set $M_{ij} = 0$ for $i < j$ and for $j < i - d$. Let the principal $(d+1) \times (d+1)$ subminor of M be an arbitrary LT invertible matrix and set $N_{ik} = \sum_{j=1}^{i} M_{ij}A_{jk}$ for $k \le i$. For $i > (d+1)$, we seek to choose M_{ij} such that N_{ij} is banded.

$$N_{i,k} = (MA)_{i,k} = M_{i,k}A_{kk} + \sum_{j=\max\{k+1,i-d\}}^{i} \sum_{\ell=1}^{d} M_{i,j}H_{j\ell}\bar{G}_{k\ell} \ . \tag{2.3}$$

N is $d+1$ banded if and only if

$$\sum_{j=\max\{1,i-d\}}^{i} \sum_{\ell=1}^{d} M_{i,j}H_{j\ell}\bar{G}_{k\ell} = 0 \quad \text{for} \quad k < i - d \ . \tag{2.4}$$

For each row $i > 2d$, (2.4) constitutes d equations for $(d+1)$ unknowns: $\sum_{j=i-d}^{i} M_{ij}H_{j\ell} = 0$. For each row $d+1 < i \le 2d$, (2.4) constitutes $i-d-1$ equations for $d+1$ unknowns. Thus there is always at least an one parameter family of solutions for each i. M is invertible if there exists a solution with $M_{ii} \neq 0$. When condition (*) is true, this requirement is satisfied. ■

This proof shows that M and N are not uniquely determined and that for $n > 2d$ the multiplicity of solutions is a degeneracy of dimension $n + d^2$ in our band fraction representation. Condition (*) is actually a stronger requirement than is necessary for the validity of Theorem 1.5. The proof requires only that (2.4) have a solution with $M_{ii} \neq 0$ while condition (*) ensures a solution under the more stringent condition that $\sum_{j=i-d}^{i} M_{ij} H_{j\ell} = 0$ for $1 \leq \ell \leq d$. This essentially corresponds to enforcing (2.4) when $i - d \leq k \leq d$.

Proof of Theorem 1.1:

We set $M_{ij} = 0$ for $i < j$ and for $j < i - d$. Let the principal $(d+1) \times (d+1)$ subminor of M be an arbitrary LT invertible matrix and set $C_{ik} = \sum_{j=1}^{i} M_{ij} B_{jk}$ for $i \leq d+1$. By hypothesis, B is LT, so C is as well. For $i > (d+1)$, we set $C_{ik} = 0$ for $k = 1 \ldots d$ and choose M_{ij} such that

$$\sum_{j=i-d}^{i} M_{ij} B_{jk} = 0 \quad \text{for} \quad 1 \leq k \leq \min\{d, i-d-1\} \tag{2.5}$$

for $d + 1 < i \leq n$. For each row i, (2.5) constitutes at most d equations for $(d+1)$ unknowns. Thus there is always at least an one parameter family of solutions for each i. M is invertible if there exists a solution with $M_{ii} \neq 0$. When condition (*) is true, this requirement is satisfied. Given M and C, we define $\tilde{N}$ as the Cholesky factor of

$$\tilde{N}\tilde{N}^* \equiv MM^* - CC^* = M\,(I - BB^*)\,M^* = MAA^*M^* \,. \tag{2.6}$$

By Theorem 4.3.1 of [3], $\tilde{N}$ has bandwidth $(d+1)$. Since $\tilde{N}$ and MA are both LT factors of MAA^*M^*, there are complex phase factor $\exp(i\theta_j)$ such that $\tilde{N}_{ij} = (MA)_{ij}\exp(i\theta_j)$. We define $N_{ij} = \tilde{N}_{ij}\exp(i\theta_j)$. ■

The construction remains valid if $C = MB$ is a band matrix of bandwidth $d+1$. Thus, condition (*) may be weakened for rows $d + 1 < i < 2d + 1$ since not all of the d subcolumns of $B_{i-d-1:i,(\cdot)}$ need to be orthogonal to the ith row of M.

3 Deriving TIB Systems from Transfer Functions

We now relate the matrix structure $A = M^{-1}N$ with M and N bidiagonal to the work of [8] on system identification in the frequency domain. Given a set of decay rates/poles of the frequency responses, $\{\lambda_n\}$, Ninness and Gustafsson derive a set of orthonormal basis functions in the frequency domain:

$$\hat{z}_n\,(w) \;=\; \sum_{t=0}^{\infty} z_n\,(t)\,w^t \;=\; w^d \frac{\sqrt{1 - |\lambda_n|^2}}{w - \lambda_n} \prod_{k=0}^{n-1} \left(\frac{1 - \lambda_k^* w}{w - \lambda_k} \right), \tag{3.1}$$

where d is 0 or 1. From this, we derive the relation:

$$\frac{w - \lambda_n}{\sqrt{1 - |\lambda_n|^2}} \hat{z}_n(w) = \frac{1 - \lambda_{n-1}^* w}{\sqrt{1 - |\lambda_{n-1}|^2}} \hat{z}_{n-1}(w) \quad . \tag{3.2}$$

Now define

$$c_n = \frac{1}{\sqrt{1 - |\lambda_n|^2}} \qquad s_n = \frac{\lambda_n}{\sqrt{1 - |\lambda_n|^2}} \quad . \tag{3.3}$$

(When λ_n is real, c_n and s_n are hyperbolic cosines and sines.) We have:

$$w \left[c_n \hat{z}_n(w) + s_{n-1}^* \hat{z}_{n-1}(w) \right] = \left[s_n \hat{z}_n(w) + c_{n-1} \hat{z}_{n-1}(w) \right] \quad . \tag{3.4}$$

Equating like powers of w yields

$$c_n z_n(t-1) + s_{n-1}^* z_{n-1}(t-1) = s_n z_n(t) + c_{n-1} z_{n-1}(t) \quad . \tag{3.5}$$

We define the semi-infinite vector of basis functions: $\mathbf{z}(t)^T = (z_0(t), z_1(t) \ldots)$. The time update formula (3.5) for the orthonormal basis functions becomes

$$M\mathbf{z} = N\mathbf{z} + \mathbf{e}_1 \quad , \tag{3.6}$$

where $\mathbf{e}_1$ is the unit vector in the first coordinate and

$$M \equiv \text{bidiag} \begin{pmatrix} c_0 & & c_1 & & c_2 & & \cdots & \\ & s_0^* & & s_1^* & & s_2^* & & \cdots \end{pmatrix} \tag{3.7}$$

$$N \equiv \text{bidiag} \begin{pmatrix} s_0 & & s_1 & & s_2 & & \cdots & \\ & c_0 & & c_1 & & c_2 & & \cdots \end{pmatrix} \quad . \tag{3.8}$$

We rewrite this update as a LTIB state space system for the basis functions:

$$\mathbf{z}(t+1) = A\mathbf{z}(t) + \mathbf{b} \; , \tag{3.9}$$

where $A = M^{-1}N$ and $\mathbf{b} = M^{-1}\mathbf{e}_1$. Define $C = \text{diag}(c_0, c_1, ...)$, we have the LU factorization

$$\begin{pmatrix} \mathbf{b} & A \end{pmatrix} = \left(M^{-1}C\right) \begin{pmatrix} C^{-1}\mathbf{e}_1 & C^{-1}N \end{pmatrix} \tag{3.10}$$

The following result gives conditions when the LU factorization of $(\mathbf{b}, A)$ may be computed stablely without partial pivoting.

Theorem 3.1 *Suppose the eigenvalues λ_k of A are arranged so that $|\lambda_k| \leq |\lambda_{k+1}|$. The LU factorization (3.10) of $(\mathbf{b}, A)$ satisfies $\max_{i,j} |L_{ij}^{-1}| = 1$ and $C^{-1}M$ is diagonally dominant.*

Proof: : Define $x_k = -s_k/c_{k+1}$. Note L is unit bidiagonal and $L_{ij}^{-1} = \prod_{j\le k<i} x_k$. The result follows when $|x_k| \le 1$. Note $|x_k|^2 = |\lambda_k|^2 \left(1 - |\lambda_{k+1}|^2\right)\left(1 - |\lambda_k|^2\right)^{-1} \le 1$. and that $|\lambda_k| \le |\lambda_{k+1}|$ implies $\left(1 - |\lambda_{k+1}|^2\right)\left(1 - |\lambda_k|^2\right)^{-1} \le 1$. Since $|\lambda_k| \le 1$ for all k we have the desired result:

$$\left(\frac{1 - |\lambda_{k+1}|^2}{1 - |\lambda_k|^2}\right) |\lambda_k|^2 \le 1. \tag{3.11}$$

■

It is not necessary to have the eigenvalues in order of increasing modulus to avoid pivoting since the condition 3.11 is always satisfied for $|\lambda_k|^2 \le \frac{1}{2}$.

3.1 Bases for Invariant Subspaces

We now evaluate the eigenvectors of the single input TIB system using the parameterization (3.6)-(3.8) for the case of distinct eigenvalues. It is well known that the eigenvectors of triangular matrices satisfy a recursion formula with $V_{ij} = 0$ for $i < j$, where V_{ij} is the i-th component of the j-th eigenvector. Write $V = [V_{ij}]$ and $\Lambda \equiv$ diag $(\lambda_1, \ldots \lambda_n)$. The eigenvector equation $AV = V\Lambda$ becomes $NV = MV\Lambda$ or element by element

$$c_k \left(c_{j-1} V_{j-1,k} + s_j V_{jk}\right) = s_k \left(s_{j-1}^* V_{j-1,k} + c_j V_{j,k}\right), \tag{3.12}$$

and solving for V_{jk}:

$$\begin{aligned} V_{jk} &= \frac{\left(c_k c_{j-1} - s_k s_{j-1}^*\right)}{\left(s_k c_j - c_k s_j\right)} V_{j-1,k} && (3.13) \\ &= \left(\frac{1 - |\lambda_j|^2}{1 - |\lambda_{j-1}|^2}\right)^{1/2} \left(\frac{1 - \lambda_k \lambda_{j-1}^*}{\lambda_k - \lambda_j}\right) V_{j-1,k} \quad , && (3.14) \end{aligned}$$

with $V_{jk} = 0$ for $j < k$. We set $V_{kk} = 1$ and solve the recursion:

$$V_{jk} = \frac{\left(1 - |\lambda_j|^2\right)^{1/2} \left(1 - |\lambda_k|^2\right)^{1/2}}{\lambda_k - \lambda_j} \left[\prod_{k<j'<j} \left(\frac{1 - \lambda_k \lambda_{j'}^*}{\lambda_k - \lambda_{j'}}\right)\right] \quad , \tag{3.15}$$

Solving for $V_{j-1,k}$ and using $c^2 - |s|^2 = 1$ yields

$$V_{j-1,k} = c s^* V_{j-1,k+1} + c^2 V_{j,k+1} \quad . \tag{3.16}$$

4 Representations if TIB Matrices

where the bracketed term is equal to 1 for $j = k + 1$. In the case of repeated eigenvalues, we work out the case of a single Jordan block with eigenvalue λ. The columns of V now satisfy $NV\mathbf{e}_k = \lambda MV\mathbf{e}_k + MV\mathbf{e}_{k+1}$ or by element:

$$c\,(cV_{j-1,k} + sV_{jk}) = s\,(s^* V_{j-1,k} + cV_{jk}) + c\,(s^* V_{j-1,k+1} + cV_{j,k+1}) \quad . \tag{4.1}$$

Solving for $V_{j-1,k}$ and using $c^2 - |s|^2 = 1$ yields

$$V_{j-1,k} = cs^* V_{j-1,k+1} + c^2 V_{j,k+1} \quad . \tag{4.2}$$

5 Representations if TIB Matrices

We now give a direct construction of the TIB filter for the rank-1 case where B is a vector $\mathbf{b}$. We define the $n \times n$ matrix $J = Z + Z^2 + \ldots Z^{n-1}$, where Z is the lower shift matrix:

$$J = \begin{pmatrix} 0 & 0 & 0 & \ldots & & \\ 1 & 0 & 0 & & & \\ 1 & 1 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix} .$$

We now specialize to invertible matrices:

Theorem 5.1 *Let A be a exponentially asymptotic stable invertible TIB matrix with $\mu = 1$; then*

$$A = \Lambda + D_b J D_g \ = \Lambda + \ \ell_{tri}(\mathbf{bg}^*), \tag{5.1}$$

where $D_b = \mathrm{diag}\{\mathbf{b}\}$, $\Lambda = \mathrm{diag}\{\lambda\}$, and $D_g = \mathrm{diag}\{\mathbf{g}\}$ are diagonal matrices. Here, $\ell_{tri}(\mathbf{bg}^*)$ denotes the strictly lower triangular part of $\mathbf{bg}^*$: $A_{k,j} = b_k \bar{g}_j, \quad k > j$. The entries satisfy

$$\lambda_j = \exp(i\theta_j) d_j \quad , \ \bar{g}_j = \exp(i\theta_j)\alpha_j \bar{b}_j / d_j \ , \tag{5.2}$$

$$d_j^2 = (1 + \alpha_j |b_j|^2) \ , \quad \alpha_j = \alpha_1 - \sum_{k=1}^{j-1} |g_k|^2 \quad , \quad \alpha_{j+1} = \alpha_j / d_j^2 \tag{5.3}$$

where $\alpha_1 = -1$ and $\exp(i\theta_j)$ is the phase of the jth eigenvalue of A: $\lambda_j = \exp(i\theta_j)|\lambda_j|$.

Proof: $(A A^*)_{i<k} = b_i \bar{b}_k \left[\alpha_i + \sum_{j<i} |g_j|^2\right] = \alpha_1 b_i \bar{b}_k$ and $(A A^*)_{k,k} = |\lambda_k|^2 + (\alpha_1 - \alpha_k)|b_k|^2 = 1 + \alpha_1 |b_k|^2$. ■

Theorem 5.1 is based on the fast update formula of Gill, Golub, Murray and Saunders [2] for the Cholesky factorization of a rank-one perturbation of a diagonal matrix. Theorem 5.1 differs from the Cholesky factor of [2] in that we have rotated the jth column by $\exp(i\theta_j)$. When $\|\mathbf{b}\| = 1$ and A is not invertible, (5.2-5.3) represent a solution to (1.1), but the solution need not be nonderogatory, nor do all solutions of (1.1) satisfy (5.2-5.3). A counterexample occurs when $\mathbf{b}$ is the unit vector, $\mathbf{e}_n$ and both J and $\mathbb{I} - \mathbf{e}_n \mathbf{e}_n^*$ satisfy (1.1).

Corollary 5.2 *Under the hypotheses of Theorem 5.1, the eigenvalues satisfy*

$$\prod_{i \le k} |\lambda_i|^2 = 1 - \sum_{i \le k} |b_i|^2 \ . \tag{5.4}$$

Proof: From Theorem 5.1, the first k eigenvalues are determined only by the first k components of $\mathbf{b}$. Equation (5.4) follows from taking determinants of the first $k \times k$ block of (1.1). ■

Corollary 5.3 *Let A satisfy the hypotheses of Theorem 5.1 with $\|\mathbf{b}\| < 1$, then A^* is upper TIB and $A^*A = \mathbb{I} - (1 - \|\mathbf{b}\|^2)\mathbf{g}\,\mathbf{g}^*$ with $\mathbf{g}$ defined in Theorem 5.1.*

Proof: $(A A^*)_{i<k} = g_i \bar{g}_k \left[|\lambda_k|^2/\alpha_k + \sum_{j>k} |b_j|^2\right] = g_i \bar{g}_k \left[-1 + \sum_{j \le k} |b_j|^2 + \sum_{j>k} |b_j|^2\right]$, where we use Corollary 5.2 to simplify $|\lambda_k|^2/\alpha_k$. Note that $|g_k|^2/\alpha_k = \alpha_k |b_k|^2/|\lambda_k|^2 = (|\lambda_k|^2 - 1)/|\lambda_k|^2$. Thus $(A A^*)_{k,k} = |\lambda_k|^2 + |g_k|^2 \sum_{j>k} |b_j|^2 = |\lambda_k|^2 [1 - |g_k|^2/\alpha_k] + |g_k|^2 \left[|\lambda_k|^2/\alpha_k + \sum_{j>k} |b_j|^2\right] = 1 - (1 - \|\mathbf{b}\|^2)|g_k|^2$. ■

When the eigenvalues of A are identical, we may simplify the representation of Theorem 5.1:

Corollary 5.4 *Under the hypotheses of Theorem 5.1, let the eigenvalues of A be identical, ($\lambda_j = \lambda$); then $|b_j| = |\lambda|^{j-1}(1 - |\lambda|^2)^{1/2}$ and $\bar{g}_j = -\lambda \bar{b}_j / |\lambda|^{2j}$. If the b_j are positive reals, then A is lower Toeplitz: $A = \text{Toeplitz}\left(\lambda, \gamma|\lambda|^{1/2}, \gamma|\lambda|, \ldots, \gamma|\lambda|^{(n-1)/2}\right)$, where $\gamma \equiv \lambda(1 - |\lambda|^2)/|\lambda|^2$.*

We can use Theorem 5.1 to construct TIB filters given the eigenvalues of $\tilde{A}$, since we can solve for $\{|b_j|\}$ given $\{\lambda_j, |\lambda_j| < 1\}$. In inverting (5.3), the complex phase of each component, b_j, is arbitrary.

Theorem 5.1 shows that $\tilde{A}\mathbf{z}$ may be computed in $3n$ flops: $(Az)_k = \lambda_k z_k + b_k \sum_{j=1}^{i-1} (\bar{g}_j z_j)$. When $\tilde{A}$ is TIB form, its inverse is as well: $\tilde{A}^{-1}\tilde{A}^{-*} = I + r\mathbf{b}\mathbf{b}^*$

where $r = (1 - \|\mathbf{b}\|^2)^{-1}$. Thus $\tilde{A}$ may be computed in O(n) operations using (5.2-5.3) with $\alpha = r$. Matrix-vector multiplication $\tilde{A}^{-1}\mathbf{z}$ is also O(n) operations.

We can replace the representation of Theorem 5.1 with a sequence of hyperbolic rotations [3].

Theorem 5.5 *Let D be a diagonal matrix with positive elements, $\gamma = \pm 1$, and let $\mathbf{b}$ be a n-vector such that $D^2 + \gamma \mathbf{b}\mathbf{b}^*$ is positive definite. The upper triangular solutions A of*

$$A A^* = D^2 + \gamma \mathbf{b}\mathbf{b}^* \tag{5.5}$$

are of the form: $(A, \mathbf{0}) = (D, \mathbf{b})G$ where $\mathbf{0}$ is a n-vector of zeros and G is S unitary matrix G (i.e. $GSG^* = S$) with $S \equiv I_n \oplus \gamma$. The matrix G is a composition of n signed Givens rotations, $G_n ... G_1$, where G_k acts only on the k^{th} column of D and $\mathbf{b}$. We define a sequence of vectors $\mathbf{b}^{(k)}$ with $\mathbf{b}^{(k-1)} = \mathbf{b}^{(k)}G_k$ with $\mathbf{b}^{(n)} \equiv \mathbf{b}$. The k^{th} rotation is determined by parameters c_k and s_k which satisfy the equations: $c_k^2 + \gamma s_k^2 = 1$ and

$$\begin{pmatrix} D_{kk} & b_k^{(k)} \end{pmatrix} \begin{pmatrix} c_k & \gamma s_k \\ -s_k & c_k \end{pmatrix} = \begin{pmatrix} A_{kk}^{(k)} & 0 \end{pmatrix} . \tag{5.6}$$

Proof: We rewrite (5.5) as

$$(D, \mathbf{b}) S (D, \mathbf{b})^* = (D, \mathbf{b}) G S G^* (D, \mathbf{b})^* = (A, \mathbf{0}) S (A, \mathbf{0})^* .$$

To obtain the upper triangular factor, we choose the k-th rotation parameters, c_k and s_k to eliminate $b_k^{(k)}$. The elements of $\mathbf{b}^{(k-1)}$ are given by

$$b_j^{(k-1)} = \begin{cases} c_k b_j^{(k)} & j < k \\ \gamma s_k D_{kk} + c_k b_k^{(k)} = 0 & j = k \\ 0 & j > k \end{cases} \tag{5.7}$$

By induction, we have $b_j^{(k)} = c_{k+1} \cdots c_d b_j$ and $A^{(k)} \equiv A^{(k+1)} G_k$ is given by

$$A_{jk}^{(k)} = \begin{cases} -s_k b_j^{(k)} & j < k \\ c_k D_{kk} - s_k b_k^{(k)} & j = k \\ 0 & j > k \end{cases} \tag{5.8}$$

■

We prefer to represent A as implicit the product of the rotation matrices. Nevertheless, the explicit representation of A is

$$A_{jk} = \begin{cases} -s_k c_{k+1} \cdots c_n b_j & j < k \\ c_k D_{kk} - s_k c_{k+1} \cdots c_n b_j & j = k \\ 0 & j > k \end{cases} . \tag{5.9}$$

6 Discussion

In system identification, the use of triangular input balanced form improves the condition of the estimate. Using Theorem 1.1, we can recast the state space advance as given in (3.6). This special structure allows for fast adaptive estimation as described in [5, 6]. The equivalence of the representation of $A = M^{-1}N$ for banded M and N and the diagonal plus strict low rank representation is a general result whose utility may extend far beyond TIB systems.

References

[1] E. ASPLUND, Inverses of matrices $\{a_{ij}\}$ which satisfy $a_{ij} = 0$ for $j > i + p$, *Math Scand.* 7:57-60 (1959).

[2] P.E. GILL, G.H. GOLUB, W. MURRAY and M.A. SAUNDERS, Methods for modifying matrix factorizations, *Math. of Comp.*, 28, (1974), pp. 505-535.

[3] G. H. GOLUB and C. F. VAN LOAN, *Matrix Computations*, third edition, John Hopkins University Press, Baltimore 1996.

[4] Y. IKEBE, On the inverse of Hessenberg matrices, *Linear Algebra and its Applications* 24:93-97 (1979).

[5] T. KAILATH and A. H. SAYED, Displacement structure: theory and applications, *SIAM Review*, 37:297-386 (1995).

[6] A. MULLHAUPT and K. S. RIEDEL, Fast identification of innovations filters, *IEEE Trans. Signal Processing* 45:2616-2619 (1997).

[7] A. MULLHAUPT and K. S. RIEDEL, Low Grade Matrices: Matrix Fraction and Consecutive Subblock Product Representations;

[8] B. NINNESS and F. GUSTAFSSON An unifying construction of orthonormal bases for system identification, manuscript.

[9] P. RÓZSA, R. BEVILACQUA, F. ROMANI, P. FAVATI, On band matrices and their inverses, *Linear Algebra and its Applications* 150:287-295 (1991).